Last time: a square matrix $B \in \mathbb{R}^{n\times n}$ has

- eigenvalues (assume distinct) $\lambda_1, \dots, \lambda_r$ given by

$$\chi_B(t) = (t-\lambda_1)^{d_1} \cdots\cdots (t-\lambda_r)^{d_r}$$

where $\underbrace{d_1, \dots, d_r}_{d_1+\dots+d_r=n}$ are the algebraic multiplicities of $\lambda_1, \dots \lambda_r$

- eigenspaces $V_{\lambda_1}, \dots, V_{\lambda_r} \subseteq \mathbb{R}^n$ which are linearly independent

whose dimensions are the geometric multiplicities of $\lambda_1, \dots \lambda_r$

$$1 \leq \dim V_{\lambda_1} \leq d_1$$
$$\vdots$$
$$1 \leq \dim V_{\lambda_r} \leq d_r$$

Today: recall that similar matrices have same spectra

$(A \sim B$ if $\exists P$ s.t. $B = PAP^{-1})$

$$\{\text{eigenvalues of } A\} = \{\text{eigenvalues of } B\}$$

THM 21.1: Moreover, the eigenspaces $V_\lambda^{(A)}$ of A for the eigenvalue λ

$\downarrow P$

are $\cong$ to the eigenspaces $V_\lambda^{(B)}$ of B for the eigenvalue λ

Upshot: similar matrices have the same eigenvalues, with the same algebraic and geometric multiplicities

Proof of this fact: $\forall$ eigenvector v of A, we have Pv is an eigenvector of B } for λ

$$Av = \lambda v \overset{\cdot P}{\Longleftrightarrow} PAv = \lambda Pv \Leftrightarrow PA\underbrace{P^{-1}P}_{I_n}v = \lambda Pv \Leftrightarrow BPv = \lambda Pv$$

Ex: B is diagonalizable $\Rightarrow B = PAP^{-1}$

where $A = \begin{pmatrix} \lambda_1 & & & & & \\ & \ddots & & & 0 & \\ & & \lambda_1 & & & \\ & & & \lambda_2 & & \\ & 0 & & & \ddots & \\ & & & & & \lambda_2 \\ & & & & & & \ddots \end{pmatrix} \begin{matrix} \}d_1 \\ \\ \}d_2 \\ \vdots \end{matrix}$ $\qquad \lambda_1 \neq \lambda_2 \neq \dots$

$$V_{\lambda_1}^{(A)} = \operatorname{span}\{e_1, e_2, \dots, e_{d_1}\}, \text{ because } A\begin{pmatrix} x_1 \\ \vdots \\ x_{d_1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \lambda_1 \begin{pmatrix} x_1 \\ \vdots \\ x_{d_1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$\downarrow \cdot P$

$$V^{(B)}_{\lambda_1} = \text{Span}\{\underbrace{Pe_1}_{\text{first column of } P}, \underbrace{Pe_2}_{\text{second column of } P}, \ldots, \underbrace{Pe_{d_1}}_{d_1\text{-th column of } P}\}$$

$\Big\}$ eigenvectors of B for λ_1

$$\left(\text{e.g. if } A = \begin{pmatrix} 2 & & & & 0 \\ & 3 & & & \\ & & 2 & & \\ & & & 2 & \\ 0 & & & & 7 \end{pmatrix}, \quad V_2 = \text{span}\{e_1, e_3, e_4\} \right)$$

New topic: Orthogonality

DEF 21.2: the inner / dot product of

$$u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \text{ and } v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n \text{ is } \quad u \cdot v = u_1 v_1 + \ldots + u_n v_n \in \mathbb{R}$$

- $$(u \cdot v) = u^T v = (u_1 \ldots u_n)\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = (u_1 v_1 + \ldots + u_n v_n)$$

- $u \cdot v = v \cdot u$

- $u \cdot (v+v') = u\cdot v + u \cdot v'$

proof: $u = \begin{pmatrix} \vdots \\ u_n \end{pmatrix}, v = \begin{pmatrix} \vdots \\ v_n \end{pmatrix}, v' = \begin{pmatrix} \vdots \\ v_n' \end{pmatrix}$

$LHS = u_1(v_1+v_1') + \ldots + u_n(v_n+v_n')$

$RHS = u_1v_1 + \ldots + u_nv_n + u_1v_1' + \ldots + u_nv_n'$

- $(u+u')\cdot v = u\cdot v + u' \cdot v$

- $u\cdot(\lambda v) = (\lambda u)\cdot v = \lambda\,(u\cdot v)$

  $\forall \lambda \in \mathbb{R}$

- $\underset{\mathbb{R}^n}{\underset{\cap}{u}} \cdot 0 = \underset{\mathbb{R}^n}{\underset{\cap}{0}} \cdot u = \underset{\mathbb{R}}{\underset{\cap}{0}}$

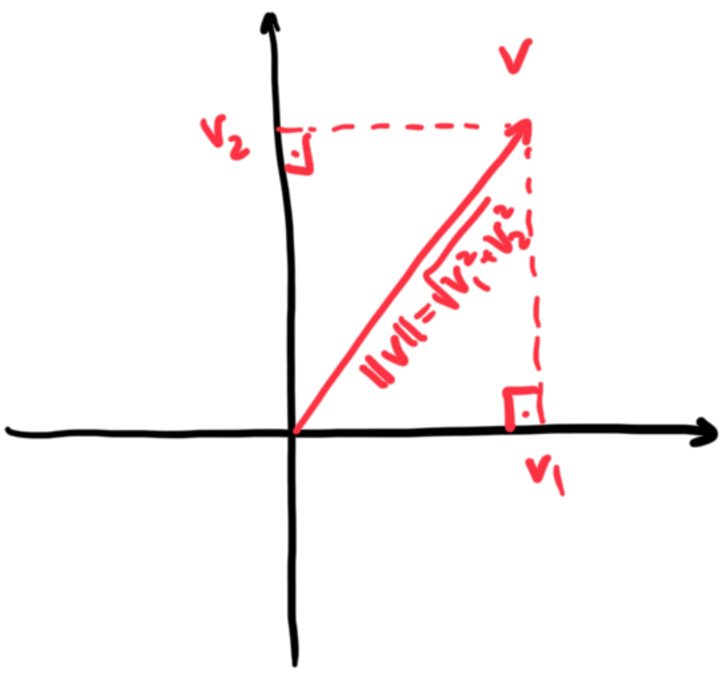
DEF & PROP 21.3

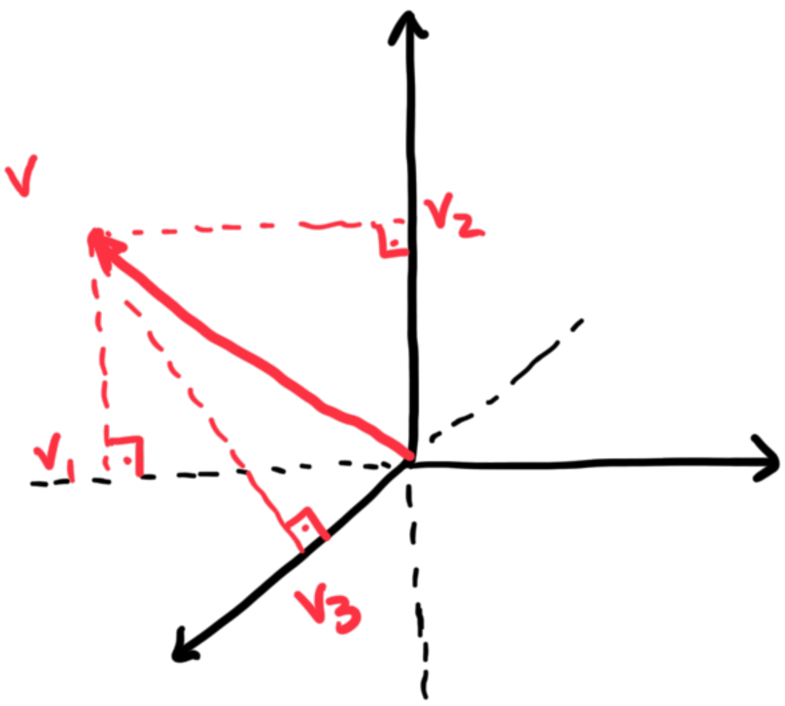
: $\forall$ vector $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n$,

we have $v\cdot v = v_1^2 + v_2^2 + \ldots\ldots + v_n^2 \geq 0$

and $\|v\| = \underset{\sqrt{v_1^2+\ldots+v_n^2}}{\underset{\shortparallel}{\sqrt{v\cdot v}}} \in \mathbb{R}_{\geq 0}$ is called the length/norm of v

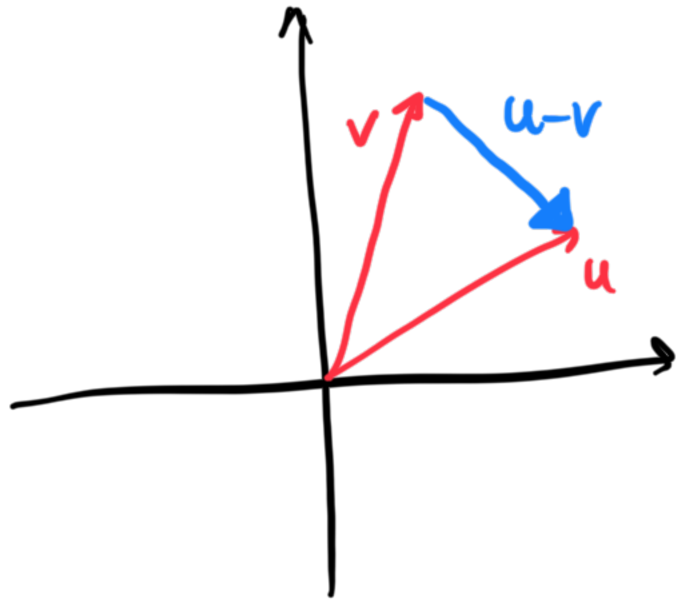
equality $\iff v = 0$





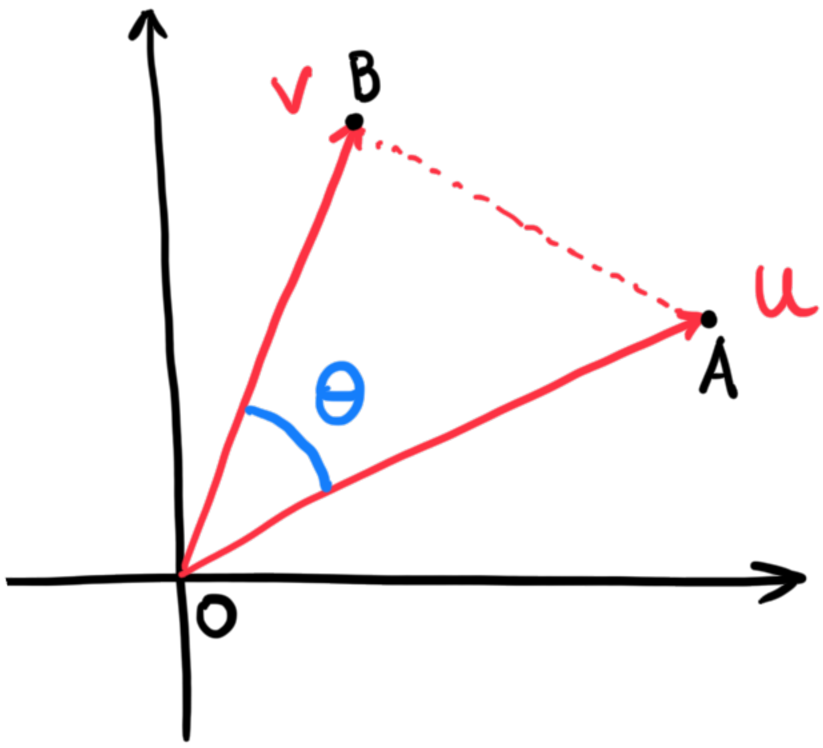
- given $u, v \in \mathbb{R}^n$, the distance between them is

$\| u - v \| \in \mathbb{R}_{\geq 0}$ $\rightsquigarrow$



Note: distance $= 0 \iff u = v$

THM 21.4: the angle θ between vectors $u, v \in \mathbb{R}^n$ satisfies $\cos\theta = \dfrac{u \cdot v}{\|u\| \|v\|}$



Proof: $OA = \sqrt{u \cdot u}$

$OB = \sqrt{v \cdot v}$

$AB = \sqrt{(u-v) \cdot (u-v)}$

Law of cosines: $AB^2 = OA^2 + OB^2 - 2 \cdot OA \cdot OB \cos\theta$

$$\cos\theta = \frac{OA^2 + OB^2 - AB^2}{2 \cdot OA \cdot OB} = \frac{u \cdot u + v \cdot v - (u-v)\cdot(u-v)}{2\|u\|\,\|v\|} =$$

$$= \frac{\cancel{u \cdot u} + \cancel{v \cdot v} - \cancel{u \cdot u} + u \cdot v + v \cdot u - \cancel{v \cdot v}}{2\|u\|\,\|v\|} = \frac{u \cdot v}{\|u\|\,\|v\|}$$

- $\cos\theta = \dfrac{u \cdot v}{\|u\|\,\|v\|}$ is unchanged by scaling u and v (up to sign)

$u \rightsquigarrow \alpha u$

$v \rightsquigarrow \beta v$

where $\alpha, \beta \in \mathbb{R}$, $\dfrac{(\alpha u)\cdot(\beta v)}{\|\alpha u\|\,\|\beta v\|} = \dfrac{\alpha\beta \quad u \cdot v}{|\alpha\beta|\,\|u\|\,\|v\|} = \operatorname{sign}(\alpha\beta)\,\dfrac{u \cdot v}{\|u\|\,\|v\|}$

upshot: making this rescaling turns

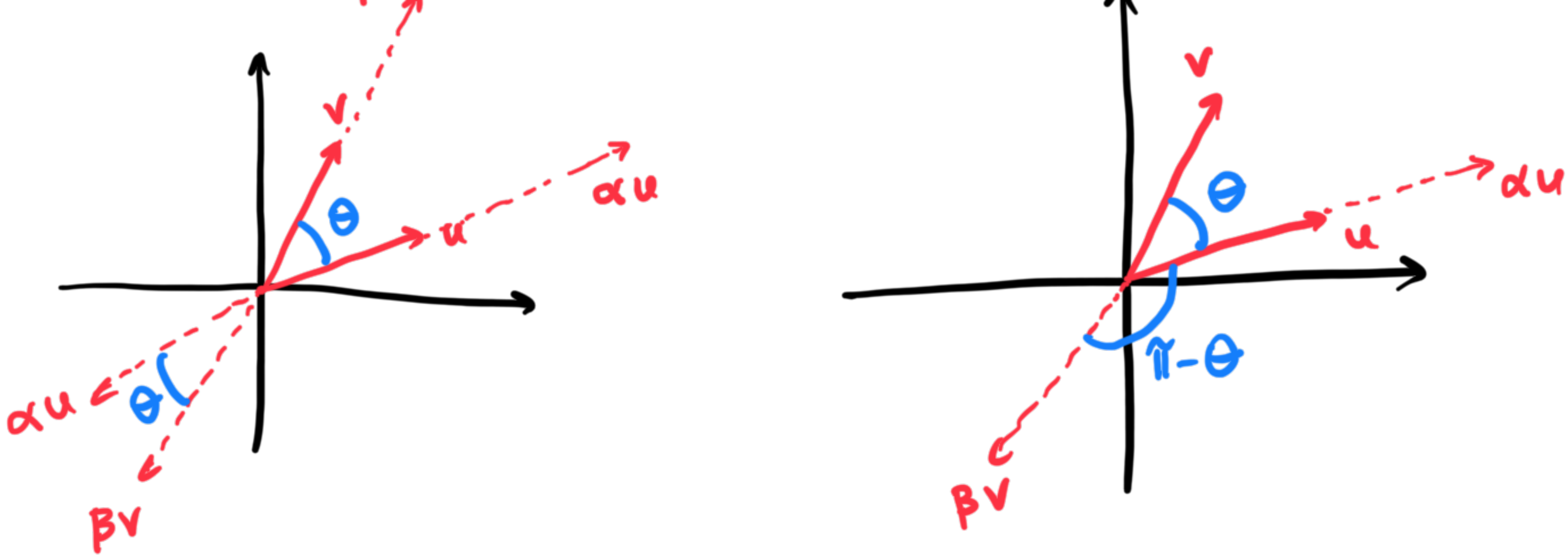
$$\cos\theta \rightsquigarrow \operatorname{sign}(\alpha\beta)\cos\theta$$

$$\Updownarrow$$

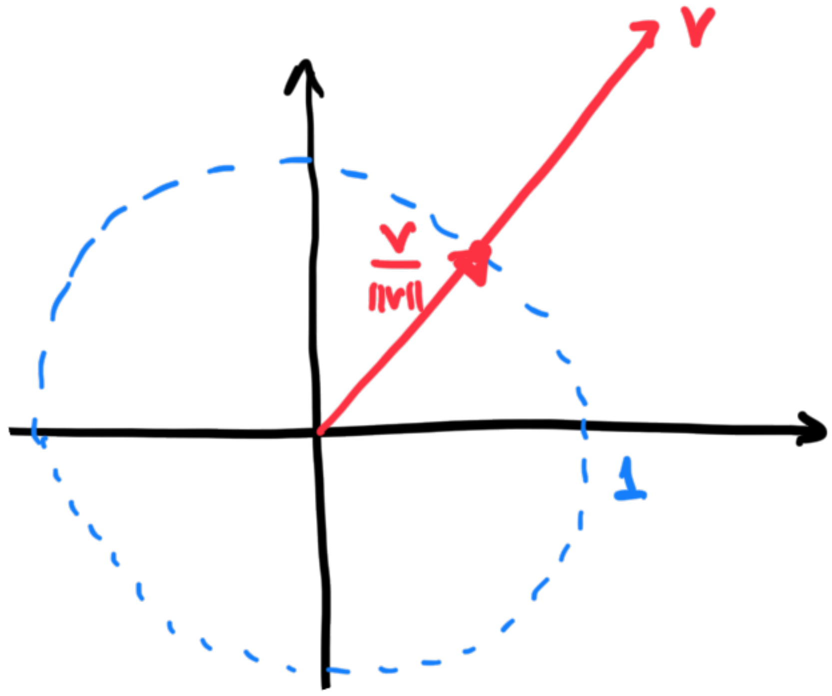
$\theta \rightsquigarrow \theta$ if $\alpha\beta > 0$ $\iff$ $\operatorname{sign}(\alpha) = \operatorname{sign}(\beta)$

$\theta \rightsquigarrow \pi - \theta$ if $\alpha\beta < 0$ $\iff$ $\operatorname{sign}(\alpha) = -\operatorname{sign}(\beta)$

βv



• $\forall$ $v \neq 0$ in $\mathbb{R}^n$, normalization / unit vector of v is



$$\frac{v}{\|v\|} \in \mathbb{R}^n$$

this whole thing has length 1, b/c $\left\| \frac{v}{\|v\|} \right\| = \frac{1}{\|v\|} \cdot \|v\| = 1$

We're using $\|\alpha v\| = \underbrace{|\alpha|}_{\in \mathbb{R}_{\geq 0}} \cdot \|v\|$

Important consequences of $\cos\theta = \dfrac{u \cdot v}{\|u\|\,\|v\|}$

• Cauchy-Schwartz inequality: $-1 \leq \cos\theta \leq 1$

$$-1 \le \frac{u \cdot v}{\|u\|\,\|v\|} \le 1$$

$$\Downarrow$$

$$-\|u\|\,\|v\| \le u \cdot v \le \|u\|\,\|v\|$$

- equality in C–S, i.e. $u \cdot v = \begin{cases} \|u\|\,\|v\| & \text{if } \cos\theta = 1 \iff \theta = 0 \\ -\|u\|\,\|v\| & \cos\theta = -1 \iff \theta = \pi \end{cases}$

$\theta = 0$ or π means u and v are collinear

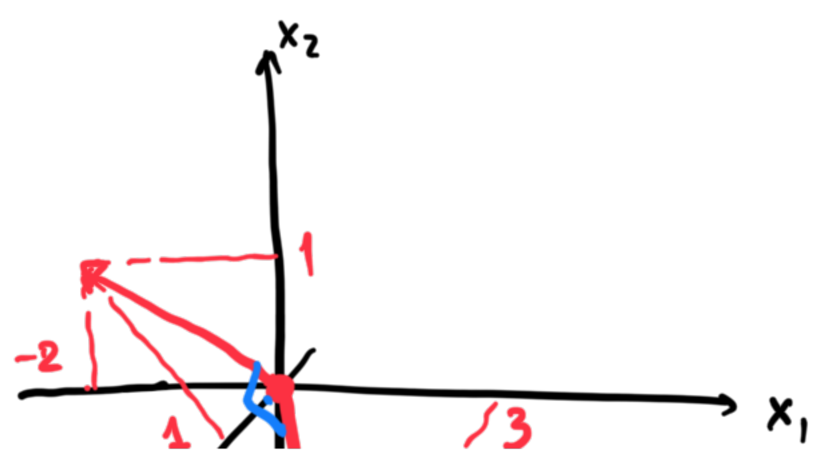
- $u \perp v$ perpendicular (orthogonality) $\iff$ $u \cdot v = 0$

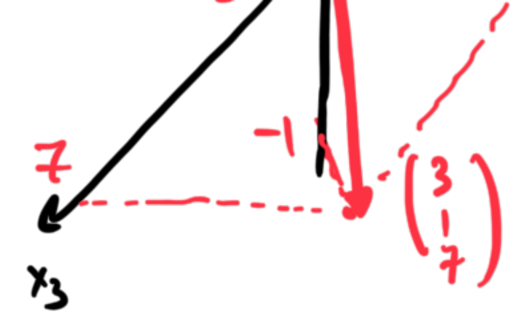
The most important consequence

Ex: prove that $\begin{pmatrix} 3 \\ -1 \\ 7 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$ are orthogonal

$$\Uparrow$$

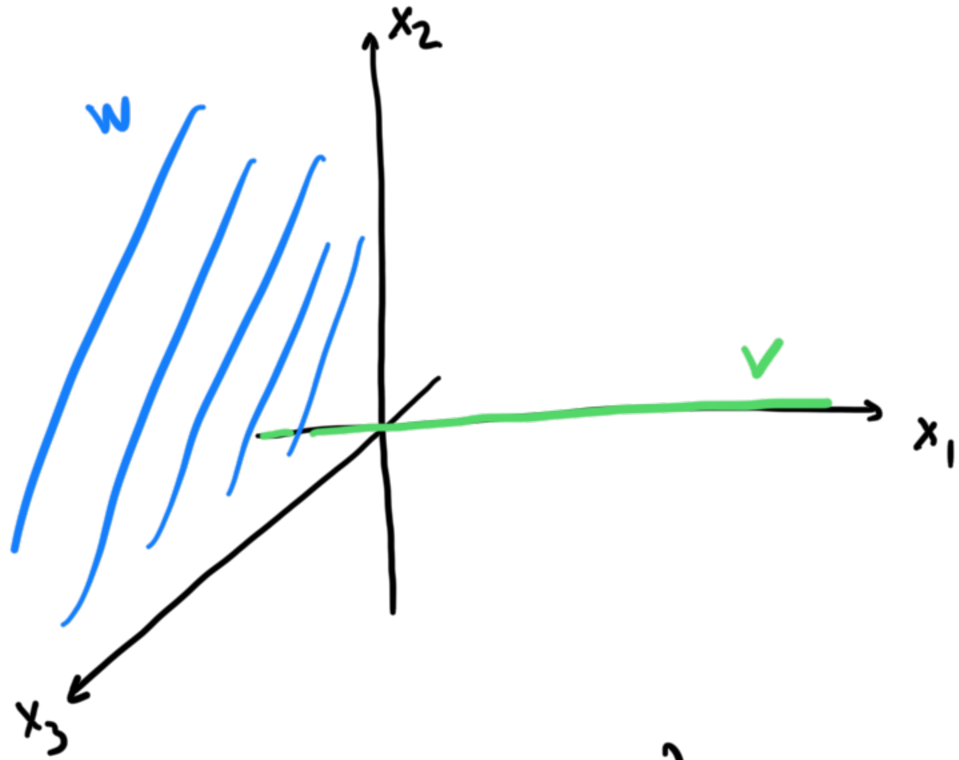
$$\begin{pmatrix} 3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 1 \end{pmatrix} = 3 \cdot (-2) + (-1) \cdot 1 + 7 \cdot 1 = 0$$





DEF 21.5 : suppose you have subspaces $V, W \subseteq \mathbb{R}^n$

we say that $V \perp W$ if $v \perp w$, $\forall v \in V$, $\forall w \in W$



$$V = \text{span}\{e_1\} = \left\{ \begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix}, \alpha \in \mathbb{R} \right\}$$

$$W = \text{span}\{e_2, e_3\} = \left\{ \begin{pmatrix} 0 \\ \beta \\ \gamma \end{pmatrix}, \beta, \gamma \in \mathbb{R} \right\}$$

$$\begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \beta \\ \gamma \end{pmatrix} = \alpha \cdot 0 + 0 \cdot \beta + 0 \cdot \gamma = 0, \quad \forall \alpha, \beta, \gamma \in \mathbb{R}$$

$$\Downarrow$$

$$V \perp W$$

THM 21.6: suppose we have subspaces $\mathbb{R}^n \supset V$ with basis $v_1, \dots, v_k$
$\mathbb{R}^n \supset W$ with basis $w_1, \dots, w_\ell$

$$V \perp W \iff v_i \perp w_j \ , \forall\ 1 \le i \le k,\ 1 \le j \le \ell$$

$$\Updownarrow$$

$$v_i \cdot w_j = 0$$

Proof: "$\Longrightarrow$" is obvious

"$\Longleftarrow$" suppose

$$\begin{array}{cccc} v_1 \cdot w_1 = 0 & v_2 \cdot w_1 = 0 & & v_k \cdot w_1 = 0 \\ v_1 \cdot w_2 = 0 & v_2 \cdot w_2 = 0 & \cdots & v_k \cdot w_2 = 0 \\ \vdots & \vdots & & \vdots \\ v_1 \cdot w_\ell = 0 & v_2 \cdot w_\ell = 0 & & v_k \cdot w_\ell = 0 \end{array}$$

we want to show that $v \cdot w = 0$, $\forall\ v \in V,\ w \in W$

$v = c_1 v_1 + \dots + c_k v_k$, $\quad w = d_1 w_1 + \dots + d_\ell w_\ell$

$$(c_1 v_1 + \dots + c_k v_k) \cdot (d_1 w_1 + \dots + d_\ell w_\ell) = \sum_{\substack{1 \le i \le k \\ 1 \le j \le \ell}} c_i d_j \underbrace{v_i \cdot w_j}_{=0} = 0 \quad \square$$

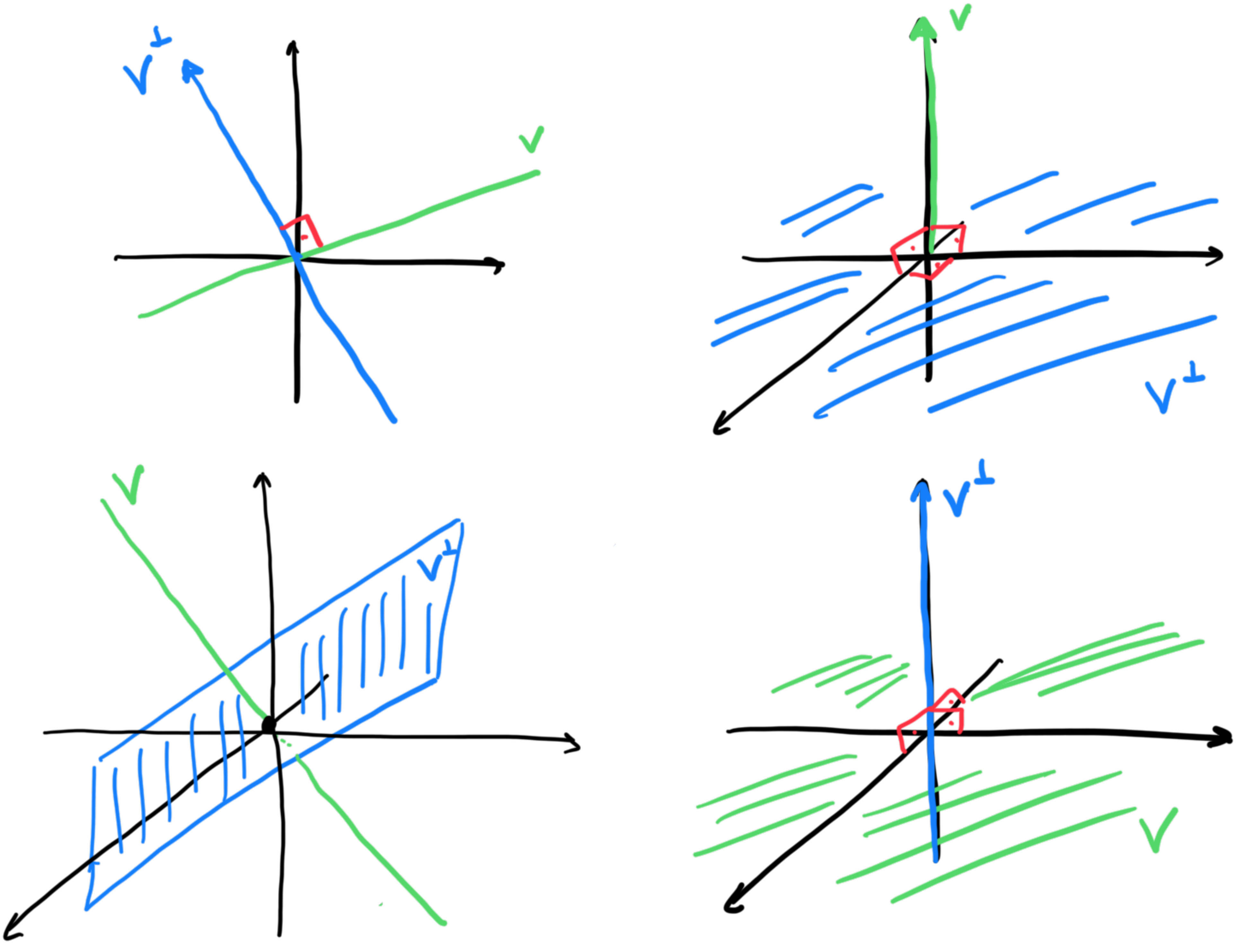
COR 21.7: if V and W are orthogonal subspaces of $\mathbb{R}^n$ (i.e. $V \perp W$), then $V \cap W = \{0\}$

Proof: if $0 \ne u \in V \cap W$, then $\begin{cases} u \in V \\ u \in W \end{cases} \Rightarrow u \cdot u = 0 \Rightarrow \|u\| = 0 \Rightarrow u = 0$

DEF 21.8: given a subspace $V \subset \mathbb{R}^n$, its orthogonal complement

$$V^\perp = \{ u \in \mathbb{R}^n \text{ s.t. } u \perp V \text{, i.e. } u \perp v, \forall v \in V \}$$

a.k.a. the biggest subspace of $\mathbb{R}^n$ which is $\perp V$



PROP 21.9: $(V^\perp)^\perp = V$

Ex : $V = \left\{ x \cdot \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + y \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \,\middle|\, x, y \in \mathbb{R} \right\} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$

$$\Downarrow$$

$$V^{\perp} = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \text{ s.t. } \underbrace{\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 0} \right\}$$

$$\begin{cases} 1 \cdot a + 2 \cdot b + 0 \cdot c = 0 \\ 0 \cdot a + 1 \cdot b + 1 \cdot c = 0 \end{cases}$$

$$= \operatorname{Ker} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Upshot (to be developed next time): $\operatorname{Col}(A)^{\perp} = \operatorname{Ker}(A^{T})$